

Super 3-Lie Algebras Induced by Super Lie Algebras

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Abstract. We propose a notion of a super n -Lie algebra and construct a super n -Lie algebra with the help of a given binary super Lie algebra which is equipped with an analog of a supertrace. We apply this approach to the super Lie algebra of a Clifford algebra with even number of generators and making use of a matrix representation of this super Lie algebra given by a supermodule of spinors we construct a series of super 3-Lie algebras labeled by positive even integers.

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1. Introduction

In search for new fundamental structures in theoretical physics and mechanics the physicists are increasingly turning their attention to algebraic structures which are based on ternary multiplication law (more generally on n -ary multiplication law). In 1973 Y. Nambu proposed a generalization of Hamiltonian mechanics where he replaced a canonically conjugate pair of variables by a triple of canonical variables and the usual Poisson bracket by a ternary operation (Nambu bracket) [4]. A geometric formalism for this generalization of Hamiltonian mechanics based on a notion of Nambu bracket of order n , the fundamental identity and a concept of Nambu-Poisson manifold was developed by L. Takhtajan in [6]. An important part of this geometric formalism is a notion of an n -Lie algebra which was also studied by V. T. Filippov in [3]. J. Arnlind, A. Makhlouf and S. Silvestrov constructed an n -Lie algebra by means of a binary Lie algebra endowed with an analog of a trace [2] and this n -Lie algebra was called an n -Lie algebra induced by Lie algebra. They also studied the cohomologies of n -Lie algebra induced by a Lie algebra and found the relation of these cohomologies to the cohomology of initial binary Lie algebra. In this paper we propose a notion of super n -Lie algebra and follow an approach of J. Arnlind, A. Makhlouf and S. Silvestrov to construct a super n -Lie algebra with the help of a binary super Lie algebra which is equipped with an analog of a supertrace,

and this super n -Lie algebra is called the super n -Lie algebra induced by a super Lie algebra. We apply this approach to the super Lie algebra of Clifford algebra (with even number of generators) with the matrix representation given by a supermodule of spinors and for any even integer $n \geq 2$ we construct a super 3-Lie algebra.

2. n -Lie algebras

In this section we remind the definition of n -Lie algebra.

Let \mathbb{K} be a field of real or complex numbers, \mathfrak{g} be a vector space over \mathbb{K} , and $\mathfrak{g}^n = \mathfrak{g} \times \mathfrak{g} \times \dots \times \mathfrak{g}$ (n times). A vector space \mathfrak{g} is said to be an n -Lie algebra if \mathfrak{g} is endowed with a multilinear skew-symmetric mapping $[\cdot, \cdot, \dots, \cdot] : \mathfrak{g}^n \rightarrow \mathfrak{g}$ which satisfies the identity

$$[y_1, y_2, \dots, y_{n-1}, [x_1, x_2, \dots, x_n]] = \sum_{k=1}^n [x_1, x_2, \dots, x_{k-1}, [y_1, y_2, \dots, y_{n-1}, x_k], x_{k+1}, \dots, x_n], \quad (2.1)$$

where $y_1, y_2, \dots, y_{n-1}, x_1, x_2, \dots, x_n \in \mathfrak{g}$. Particularly if $n = 2$ then the above definition yields the definition of a (binary) Lie algebra and the identity (2.1) takes on the form of Jacobi identity. Let \mathfrak{h} be a (binary) Lie algebra with a Lie bracket $[\cdot, \cdot] : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathfrak{h}$, and $\phi : \mathfrak{h}^k \rightarrow \mathbb{K}, k \geq 1$ be an \mathbb{K} -multilinear skew-symmetric k -form which satisfies the condition

$$\sum_{m=1}^k \phi(x_1, x_2, \dots, x_{m-1}, [x_m, y], x_{m+1}, \dots, x_k) = 0, \quad (2.2)$$

where x_1, x_2, \dots, x_k, y are elements of \mathfrak{h} . Particularly if $k = 1$ then a form $\phi : \mathfrak{h} \rightarrow \mathbb{K}$ satisfies $\phi([x, y]) = 0$ for any $x, y \in \mathfrak{h}$.

Theorem 2.1. *Let $k \geq 1$ be an integer, $\sigma = (i_1, i_2, \dots, i_k, i_{k+1}, i_{k+2})$ be a permutation of the integers $(1, 2, \dots, k+2)$ such that $1 \leq i_1 < i_2, \dots < i_k \leq k+2$, $1 \leq i_{k+1} < i_{k+2} \leq k+2$ and $|\sigma|$ be the parity of σ . Define*

$$[x_1, x_2, \dots, x_{k+2}] = \sum_{\sigma} (-1)^{|\sigma|} \phi(x_{i_1}, x_{i_2}, \dots, x_{i_k}) [x_{i_{k+1}}, x_{i_{k+2}}]. \quad (2.3)$$

Then \mathfrak{h} endowed with (2.3) is the n -Lie algebra, where $n = k+2$.

It is worth noting that in the particular case of $k = 1$ this theorem yields the 3-Lie algebra with ternary Lie bracket

$$[x, y, z] = \phi(x)[y, z] + \phi(y)[z, x] + \phi(z)[x, y], \quad (2.4)$$

and a form $\phi : \mathfrak{h} \rightarrow \mathbb{K}$ which for any $x, y \in \mathfrak{h}$ satisfies $\phi([x, y]) = 0$ can be viewed as an analog of a trace. The 3-Lie algebras of this kind induced by a binary Lie algebra are introduced and studied [?].

3. Super n -Lie algebras

In this section we give a definition of a super n -Lie algebra and prove that one can construct a super n -Lie algebra by means of a super trace.

Let $V = V_0 \oplus V_1$ be a finite-dimensional super vector space, where V_0 is the subspace of even elements and V_1 is the subspace of odd elements. If $v \in V$ is a homogeneous element then its degree will be denoted by $|v|$, where $|v| \in \mathbb{Z}_2$ and $\mathbb{Z}_2 = \{0, 1\}$. Let $V^n = V \times V \times \dots \times V$. For any $\mathbf{v} = (v_1, v_2, \dots, v_n) \in V^n$, where v_1, v_2, \dots, v_n are homogeneous elements, and for every integer $k = 1, 2, \dots, n$ we define

$$|\mathbf{v}|_k = \sum_{i=1}^k |v_i|.$$

Let $\text{End}(V)$ be the super vector space of endomorphisms of a super vector space V . The composition of two endomorphisms $a \circ b$ determines the structure of superalgebra in $\text{End}(V)$, and the graded binary commutator $[a, b] = a \circ b - (-1)^{|a||b|} b \circ a$ induces the structure of super Lie algebra in $\text{End}(V)$. The supertrace of an endomorphism $a : V \rightarrow V$ can be defined by

$$\text{Str}(a) = \begin{cases} \text{Tr}(a|_{V_0}) - \text{Tr}(a|_{V_1}) & \text{if } a \text{ is even,} \\ 0 & \text{if } a \text{ is odd.} \end{cases} \quad (3.1)$$

For any endomorphisms a, b it holds $\text{Str}([a, b]) = 0$.

Given $\mathbf{v} = (v_1, v_2, \dots, v_n) \in V^n$ and a permutation $\sigma = (i_1, i_2, \dots, i_n)$ of the integers $(1, 2, \dots, n)$ we assign to them the element $\mathbf{v}_\sigma \in V^n$ defined by

$$(\mathbf{v}, \sigma) \rightarrow \mathbf{v}_\sigma = (v_{i_1}, v_{i_2}, \dots, v_{i_n}).$$

Let i_k be an element of permutation σ . Assume that $(i_{k_1}, i_{k_2}, \dots, i_{k_r})$ are the elements of permutation σ which precede an element i_k in σ and greater than i_k . Obviously r is the number of inversions of an element i_k in σ . Define the integer $|\mathbf{v}_\sigma| \in \mathbb{Z}_2$ by

$$|\mathbf{v}_\sigma| = \sum_{k=1}^n |v_{i_k}| (|v_{i_{k_1}}| + |v_{i_{k_2}}| + \dots + |v_{i_{k_r}}|).$$

Definition 3.1. A super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is said to be a super n -Lie algebra if it is endowed with an n -ary bracket $[\cdot, \cdot, \dots, \cdot] : \mathfrak{g} \times \mathfrak{g} \times \dots \times \mathfrak{g} (n \text{ times}) \rightarrow \mathfrak{g}$ which satisfies the following conditions:

1. n -ary bracket is a multilinear mapping and the degree of n -ary bracket of n homogeneous elements is equal to the sum of degrees of these elements, i.e. for any homogeneous element $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathfrak{g}^n$ it holds

$$|[x_1, x_2, \dots, x_n]| = |\mathbf{x}|_n,$$

2. n -ary bracket is a graded skew-symmetric multilinear mapping, i.e. for any homogeneous elements $x_1, x_2, \dots, x_n \in \mathfrak{g}$ and for any $k = 1, 2, \dots, n-1$ it holds

$$[x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_n] = -(-1)^{|x_k||x_{k+1}|} [x_1, x_2, \dots, x_{k+1}, x_k, \dots, x_n],$$

3. n -ary bracket satisfies the identity

$$[y_1, y_2, \dots, y_{n-1}, [x_1, x_2, \dots, x_n]] = \sum_{k=1}^n (-1)^{|\mathbf{x}|_{k-1} |\mathbf{y}|_{n-1}} [x_1, x_2, \dots, x_{k-1}, [y_1, y_2, \dots, y_{n-1}, x_k], x_{k+1}, \dots, x_n],$$

where $\mathbf{y} = (y_1, y_2, \dots, y_{n-1}) \in \mathfrak{g}^{n-1}$, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathfrak{g}^n$ are homogeneous elements.

Lemma 3.2. *Let $\text{End}(V)$ be the super vector space of endomorphisms of a super vector space V and $\mathbf{a} = (a_1, a_2, \dots, a_n)$ be a sequence of n endomorphisms of V . Define*

$$[a_1, a_2, \dots, a_n] = \sum_{\sigma} (-1)^{|\sigma| + |\mathbf{a}_{\sigma}|} a_{i_1} \circ a_{i_2} \circ \dots \circ a_{i_n}, \quad (3.2)$$

where $\sigma = (i_1, i_2, \dots, i_n)$ is a permutation of integers $(1, 2, \dots, n)$ and $|\sigma|$ is the parity of this permutation. Then the super vector space $\text{End}(V)$ endowed with the n -ary bracket (3.2) is the super n -Lie algebra.

Definition 3.3. A representation of a super n -Lie algebra \mathfrak{g} is a linear mapping $\rho : \mathfrak{g} \rightarrow \text{End}(V)$, where V is a super vector space (a representation space of \mathfrak{g}), which satisfies:

1. for any homogeneous element $x \in \mathfrak{g}$ its image $\rho(x)$ in $\text{End}(V)$ is the homogeneous element and $|\rho(x)| = |x|$,
2. for any elements $x_1, x_2, \dots, x_n \in \mathfrak{g}$ it holds

$$\rho([x_1, x_2, \dots, x_n]) = [\rho(x_1), \rho(x_2), \dots, \rho(x_n)].$$

Let \mathfrak{g} be a super n -Lie algebra with n -ary bracket $[\cdot, \cdot, \dots, \cdot]$, and $\rho : \mathfrak{g} \rightarrow \text{End}(V)$ be a representation of \mathfrak{g} . For any homogeneous element $\mathbf{x} = (x_1, x_2, \dots, x_{n+1})$ of \mathfrak{g} we define the $(n+1)$ -ary bracket by

$$[x_1, x_2, \dots, x_{n+1}]_{\rho} = \sum_{k=1}^{n+1} (-1)^{k-1} (-1)^{|x_k| |\mathbf{x}|_{k-1}} \text{Str}(\rho(x_k)) \times [x_1, x_2, \dots, \hat{x}_k, \dots, x_{n+1}], \quad (3.3)$$

where the hat over an element \hat{x}_k means that this element is omitted.

Theorem 3.4. *A super n -Lie algebra \mathfrak{g} equipped with the $(n+1)$ -ary bracket (3.3) is the super $(n+1)$ -ary Lie algebra.*

4. Supermodule over Clifford algebra

A Clifford algebra is a unital associative algebra, which can be equipped with an \mathbb{Z}_2 -graded structure, and it provides a well known example of a superalgebra. Taking the graded commutator of two elements of this superalgebra one can consider it as the super Lie algebra. In this section we consider a supermodule over Clifford algebra with even number of generators, and this supermodule provides us with a representation of mentioned above super Lie algebra. Making use of a supertrace of

this representation we construct a super 3-Lie algebra with the help of the formula (3.3).

We remind that Clifford algebra C_n is the unital associative algebra over \mathbb{C} generated by $\gamma_1, \gamma_2, \dots, \gamma_n$ which obey the relations

$$\gamma_i \gamma_j + \gamma_j \gamma_i = 2 \delta_{ij} e, \quad i, j = 1, 2, \dots, n, \quad (4.1)$$

where e is the unit element of Clifford algebra. Let $\mathcal{N} = \{1, 2, \dots, n\}$ be the set of integers from 1 to n . If I is a subset of \mathcal{N} , i.e. $I = \{i_1, i_2, \dots, i_k\}$ where $1 \leq i_1 < i_2 < \dots < i_k \leq n$, then one can associate to this subset I the monomial $\gamma_I = \gamma_{i_1} \gamma_{i_2} \dots \gamma_{i_k}$. If $I = \emptyset$ one defines $\gamma_\emptyset = e$. The number of elements of a subset I will be denoted by $|I|$. It is obvious that the vector space of Clifford algebra C_n is spanned by the monomials γ_I , where $I \subseteq \mathcal{N}$. Hence the dimension of this vector space is 2^n and any element $x \in C_n$ can be expressed in terms of these monomials as

$$x = \sum_{I \subseteq \mathcal{N}} a_I \gamma_I,$$

where $a_I = a_{i_1 i_2 \dots i_k}$ is a complex number. It is easy to see that one can endow a Clifford algebra C_n with the \mathbb{Z}_2 -graded structure by assigning the degree $|\gamma_I| = |I| \pmod{2}$ to monomial γ_I . Then a Clifford algebra C_n can be considered as the superalgebra since for any two monomials it holds $|\gamma_I \gamma_J| = |\gamma_I| + |\gamma_J|$.

Another way to construct this superalgebra which does not contain explicit reference to Clifford algebra is given by the following theorem.

Theorem 4.1. *Let I be a subset of $\mathcal{N} = \{1, 2, \dots, n\}$, and γ_I be a symbol associated to I . Let C_n be the vector space spanned by the symbols γ_I . Define the degree of γ_I by $|\gamma_I| = |I| \pmod{2}$, where $|I|$ is the number of elements of I , and the product of γ_I, γ_J by*

$$\gamma_I \gamma_J = (-1)^{\sigma(I, J)} \gamma_{I \Delta J}, \quad (4.2)$$

where $\sigma(I, J) = \sum_{j \in J} \sigma(I, j)$, $\sigma(I, j)$ is the number of elements of I which are greater than $j \in J$, and $I \Delta J$ is the symmetric difference of two subsets. Then C_n is the unital associative superalgebra, where the unit element e is γ_\emptyset .

This theorem can be proved by means of the properties of symmetric difference of two subsets. We remind a reader that the symmetric difference is commutative $I \oplus J = J \oplus I$, associative $(I \Delta J) \Delta K = I \Delta (J \Delta K)$ and $I \Delta \emptyset = \emptyset \Delta I$. The latter shows that γ_\emptyset is the unit element of this superalgebra. The symmetric difference also satisfies $|I \Delta J| = |I| + |J| \pmod{2}$. Hence C_n is the superalgebra.

The superalgebra C_n can be considered as the super Lie algebra if for any two homogeneous elements x, y of this superalgebra one introduces the graded commutator $[x, y] = xy - (-1)^{|x||y|}yx$ and extends it by linearity to a whole superalgebra C_n . We will denote this super Lie algebra by \mathfrak{C}_n . Then $\{\gamma_I\}_{I \subseteq \mathcal{N}}$ are the generators of this super Lie algebra \mathfrak{C}_n , and its structure is entirely determined by the graded commutators of γ_I . Then for any two generators γ_I, γ_J we have

$$[\gamma_I, \gamma_J] = f(I, J) \gamma_{I \Delta J}, \quad (4.3)$$

where $f(I, J)$ is the integer-valued function of two subsets of \mathcal{N} defined by

$$f(I, J) = (-1)^{\sigma(I, J)} (1 - (-1)^{|I \cap J|}),$$

It is easy to verify that the degree of graded commutator is consistent with the degrees of generators, i.e. $[\gamma_I, \gamma_J] = |\gamma_I| + |\gamma_J|$. Indeed the function $\sigma(I, J)$ satisfies

$$\sigma(I, J) = |I||J| - |I \cap J| - \sigma(I, J),$$

and

$$\begin{aligned} f(J, I) &= (-1)^{\sigma(J, I)} (1 - (-1)^{|I \cap J|}) \\ &= (-1)^{|I||J| - |I \cap J| - \sigma(I, J)} (1 - (-1)^{|I \cap J|}) \\ &= (-1)^{|I||J|} (-1)^{\sigma(I, J)} ((-1)^{|I \cap J|} - 1) = -(-1)^{|I||J|} f(I, J). \end{aligned}$$

Hence $[\gamma_I, \gamma_J] = -(-1)^{|I||J|} [\gamma_J, \gamma_I]$ which shows that the relation (4.3) is consistent with the symmetries of graded commutator. It is obvious that if the intersection of subsets I, J contains an even number of elements then $f(I, J) = 0$, and the graded commutator of γ_I, γ_J is trivial. Particularly if at least one of two subsets I, J is the empty set then $f(I, J) = 0$. Thus any graded commutator (4.3) containing e is trivial.

As an example, consider the super Lie algebra \mathfrak{C}_2 . Its underlying vector space is 4-dimensional and \mathfrak{C}_2 is generated by two even degree generators e, γ_{12} and two odd degree generators γ_1, γ_2 . The non-trivial relations of this super Lie algebra are given by

$$[\gamma_1, \gamma_1] = [\gamma_2, \gamma_2] = 2e, \quad [\gamma_1, \gamma_{12}] = 2\gamma_2, \quad [\gamma_2, \gamma_{12}] = -2\gamma_1. \quad (4.4)$$

Now we assume that $n = 2m, m \geq 1$ is an even integer. The super Lie algebra \mathfrak{C}_n has a matrix representation which can be described as follows. Fix $n = 2$ and identify the generators γ_1, γ_2 with the Pauli matrices σ_1, σ_2 , i.e.

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \quad (4.5)$$

Then $\gamma_{12} = \gamma_1 \gamma_2 = i \sigma_3$ where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let S^2 be the 2-dimensional complex super vector space \mathbb{C}^2 with the odd degree operators (4.5), where the \mathbb{Z}_2 -graded structure of S^2 is determined by $\sigma_3 = i^{-1} \gamma_{12}$. Then $C_2 \simeq \text{End}(S^2)$, and S^2 can be considered as a supermodule over the superalgebra C_2 . Let $S^n = S^2 \otimes S^2 \otimes \dots \otimes S^2$ (m - times). Then S^n can be viewed as a supermodule over the m -fold tensor product of C_2 , which can be identified with C_n by identifying γ_1, γ_2 in the j th factor with $\gamma_{2j-1}, \gamma_{2j}$ in C_n . This C_n -supermodule S^n is called the supermodule of spinors [5]. Hence we have the matrix representation for the Clifford algebra C_n , and this matrix representation or supermodule of spinors allows one to consider the supertrace, and it can be proved [5] that

$$\text{Str}(\gamma_I) = \begin{cases} 0 & \text{if } I < \mathcal{N}, \\ (2i)^m & \text{if } I = \mathcal{N}. \end{cases} \quad (4.6)$$

Now we have the super Lie algebra \mathfrak{C}_n with the graded commutator defined in (4.3) and its matrix representation based on the supermodule of spinors. Hence we can construct a super 3-Lie algebra by making use of graded ternary commutator (3.3).

Applying the formula (3.3) we define the graded ternary commutator for any triple $\gamma_I, \gamma_J, \gamma_K$ of elements of basis for \mathfrak{C}_n by

$$[\gamma_I, \gamma_J, \gamma_K] = \text{Str}(\gamma_I) [\gamma_J, \gamma_K] - (-1)^{|I||J|} \text{Str}(\gamma_J) [\gamma_I, \gamma_K] + (-1)^{|K|(|I|+|J|)} \text{Str}(\gamma_K) [\gamma_I, \gamma_J], \quad (4.7)$$

where the binary graded commutator at the right-hand side of this formula is defined by (4.3). According to Theorem 3.4 the vector space spanned by $\gamma_I, I \subset \mathcal{N}$ and equipped with the ternary graded commutator (4.7) is the super 3-Lie algebra which will be denoted by $\mathfrak{C}_n^{(3)}$. Making use of (4.3) we can write the expression at the right-hand side of the above formula in the form

$$[\gamma_I, \gamma_J, \gamma_K] = f(J, K) \text{Str}(\gamma_I) \gamma_{J \Delta K} - (-1)^{|I||J|} f(I, K) \text{Str}(\gamma_J) \gamma_{I \Delta K} + (-1)^{|K|(|I|+|J|)} f(I, J) \text{Str}(\gamma_K) \gamma_{I \Delta J}.$$

From the formula for supertrace (4.6) it follows immediately that the above graded ternary commutator is trivial if none of subsets $\gamma_i, \gamma_J, \gamma_K$ is equal to \mathcal{N} . Similarly this graded ternary commutator is also trivial if all three subsets I, J, K are equal to \mathcal{N} , i.e. $I = J = K = \mathcal{N}$, or two of them are equal to \mathcal{N} .

Theorem 4.2. *The graded ternary commutators of the generators $\gamma_I, I \subseteq \mathcal{N}$ of the super 3-Lie algebra $\mathfrak{C}_n^{(3)}$ are given by*

$$[\gamma_I, \gamma_J, \gamma_K] = \begin{cases} (2i)^m f(I, J) \gamma_{I \Delta J} & \text{if } I \neq \mathcal{N}, J \neq \mathcal{N}, K = \mathcal{N}, \\ 0 & \text{in all other cases.} \end{cases} \quad (4.8)$$

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